# Suggested Solution to Assignment 5

1. (a) For coalition $S=\{A_1,A_2\}$, we compute $v(S)$ and $v(S^c)$ as follows. First the game bimatrix for the 2-person game between $S$ and $S^c$ is $\begin{pmatrix} (1,5) & (-1,7) \\ (6,0) & (7,-1) \\ (-1,7) & (0,6) \\ (8,-2) & (4,2) \end{pmatrix}$.

Now the payoff matrix for $S=\{A_1,A_2\}$ is $\begin{pmatrix} 1 & -1 \\ 6 & 7 \\ -1 & 0 \\ 8 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 6 & 7 \\ 8 & 4 \end{pmatrix}$

The value of $v(\{A_1,A_2\})$ is equal to the value of the above game matrix which is equal to

$\frac{6\times4-8\times7}{6+4-8-7}=\frac{32}{5}$ Note that the sum of the payoffs to $S$ and $S^c$ is always equal to 6, hence $v(\{A_3\})=-\frac{2}{5}$. Obviously, consider coalition $S=\{A_1,A_2,A_3\}$, we have $v(\{A_1,A_2,A_3\})=6$.

Similarly, consider coalition $S=\{A_1,A_3\}$, we compute $v(\{A_1,A_3\})=8$ and $v(\{A_2\})=-2$.

And consider coalition $S=\{A_2,A_3\}$, we compute $v(\{A_2,A_3\})=\frac{15}{2}$ and $v(\{A_1\})=-\frac{3}{2}$.

Therefore, the characteristic function of the game is $v(\{A_1\})=-\frac{3}{2}$, $v(\{A_2\})=-\frac{3}{2}$, $v(\{A_3\})=-\frac{2}{5}$, $v(\{A_1,A_2\})=\frac{32}{5}$, $v(\{A_1,A_3\})=\frac{15}{2}$, $v(\{A_2,A_3\})=\frac{15}{2}$, $v(\{A_1,A_2,A_3\})=6$.

(b) Pf. Let $\vec{x}=(x_1,x_2,x_3)\in I(v)$ be an imputation. Then $\vec{x}\in C(v)$ if and only if

$$\begin{cases} x_1\geq-\frac{3}{2},\ x_2\geq-2,\ x_3\geq-\frac{2}{5} \\ x_1+x_2\geq\frac{32}{5},\ x_1+x_3\geq 8,\ x_2+x_3\geq\frac{15}{2} \\ x_1+x_2+x_3=v(A)=6 \end{cases}$$

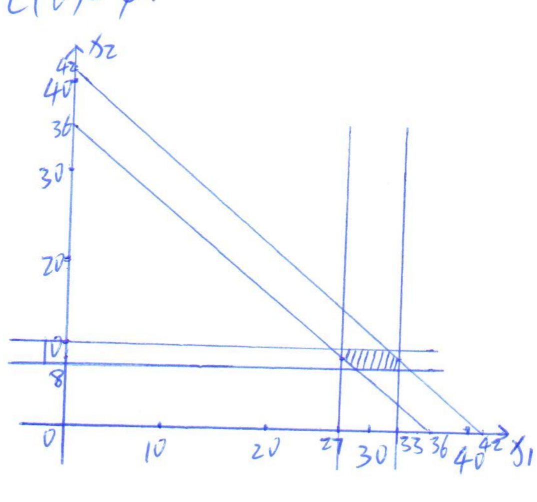
Now
$$-\frac{3}{2}\leq x_1\leq 6-x_2-x_3\leq 6-\frac{15}{2}=-\frac{3}{2} \Rightarrow x_1=-\frac{3}{2}$$
$$-2\leq x_2\leq 6-x_1-x_3\leq 6-8=-2 \Rightarrow x_2=-2$$
$$-\frac{2}{5}\leq x_3\leq 6-x_1-x_2\leq 6-\frac{32}{5}=-\frac{2}{5} \Rightarrow x_3=-\frac{2}{5}$$
$$\Rightarrow x_1+x_2+x_3=-\frac{39}{10}$$

which contradicts with $x_1+x_2+x_3=6$. Therefore $C(v)=\emptyset$.

2. Let $\vec{x}=(x_1,x_2,x_3)\in C(v)$ if and only if

$$\begin{cases} x_1\geq 27,\ x_2\geq 8,\ x_3\geq 18 \\ x_1+x_2\geq 36,\ x_1+x_3\geq 50,\ x_2+x_3\geq 27 \\ x_1+x_2+x_3=60 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 27\leq x_1\leq 33 \\ 8\leq x_2\leq 10 \\ 18\leq x_3\leq 24 \\ x_1+x_2+x_3=60 \end{cases} \xLeftrightarrow{x_3=60-x_1-x_2} \begin{cases} 27\leq x_1\leq 33 \\ 8\leq x_2\leq 10 \\ 36\leq x_1+x_2\leq 42 \end{cases}$$




shadow region is core $C(v)$.

3. (a) First we have $u(\{1\})=u(\{2\})=u(\{3\})=0$, $u(\{1,2,3\})=1$.

Next we calculate $k=\dfrac{1}{v(\{1,2,3\})-(v(\{1\})+v(\{2\})+v(\{3\}))}=\dfrac{1}{20-(3+4+6)}=\dfrac{1}{7}$

and we have

$$u(\{1,2\})=k(v(\{1,2\})-(v(\{1\})+v(\{2\})))=\frac{1}{7}(9-(3+4))=\frac{2}{7}$$

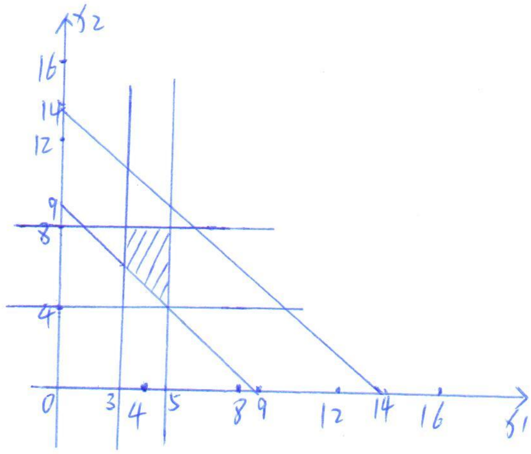
$$u(\{1,3\})=k(v(\{1,3\})-(v(\{1\})+v(\{3\})))=\frac{1}{7}(12-(3+6))=\frac{3}{7}$$

$$u(\{2,3\})=k(v(\{2,3\})-(v(\{2\})+v(\{3\})))=\frac{1}{7}(15-(4+6))=\frac{5}{7}.$$

(b) Let $\vec{x}=(x_1,x_2,x_3)\in C(v)$ if and only if

$$\begin{cases} x_1\geq 3,\ x_2\geq 4,\ x_3\geq 6 \\ x_1+x_2\geq 9,\ x_1+x_3\geq 12,\ x_2+x_3\geq 15 \\ x_1+x_2+x_3=20 \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} 3\leq x_1\leq 5 \\ 4\leq x_2\leq 8 \\ 9\leq x_1+x_2\leq 14 \end{cases}$$




Shadow region is core $C(v)$.

(c) To find the Shapley value $\phi_1$ of 1, observe that the coalitions containing 1 are $\{1\}$, $\{1,2\}$, $\{1,3\}$ and $\{1,2,3\}$. Thus

$$\phi_1=\frac{(3-1)!(1-1)!}{3!}(v(\{1\})-v(\emptyset))+\frac{(3-2)!(2-1)!}{3!}(v(\{1,2\})-v(\{2\}))+\frac{(3-2)!(2-1)!}{3!}(v(\{1,3\})-v(\{3\}))$$
$$+\frac{(3-3)!(3-1)!}{3!}(v(\{1,2,3\})-v(\{2,3\}))$$
$$=\frac{2}{6}(3)+\frac{1}{6}(9-4)+\frac{1}{6}(12-6)+\frac{2}{6}(20-15)$$
$$=\frac{9}{2}.$$

Similarly, we have $\phi_2=\frac{13}{2}$ and $\phi_3=9$.

4. By the conditions, we have

$v(\{A\})=v(\{B\})=v(\{C\})=0$, $v(\{A,B\})=11+7-15=3$, $v(\{A,C\})=11+8-14=5$,

$v(\{B,C\})=7+8-13=2$, $v(\{A,B,C\})=11+7+8-20=6$.

Since $v(\{A\})=v(\{B\})=v(\{C\})=0$, we may use the formula

$$\phi_1=\frac{2v(\{A,B,C\})+v(\{A,B\})+v(\{A,C\})-2v(\{B,C\})}{6}=\frac{2\times 6+3+5-2\times 2}{6}=\frac{8}{3}.$$

$$\phi_2=\frac{1}{6}(2v(\{A,B,C\})+v(\{A,B\})+v(\{B,C\})-2v(\{A,C\}))=\frac{1}{6}(2\times 6+3+2-2\times 5)=\frac{7}{6}.$$

$\phi_3 = \frac{1}{6}(2\times 6 + 5 + 2 - 2\times 3) = \frac{13}{6}.$

If they cooperate, the amount that each of them should pay is

A: $11 - \phi_1 = 11 - \frac{8}{3} = \frac{25}{3}$ ;

B: $7 - \phi_2 = 7 - \frac{7}{6} = \frac{35}{6}$ ;

C: $8 - \phi_3 = 8 - \frac{13}{6} = \frac{35}{6}$ .

5. By the conditions, we have $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = 0$ and

$$\begin{cases} v(\{1,2\}) = v(\{1,3\}) = v(\{1,4\}) = 1 \\ v(\{2,3\}) = v(\{2,4\}) = v(\{3,4\}) = 0 \\ v(S) = 1 \text{ for any } S \text{ with } |S| \geq 3 \end{cases}$$

Thus $\phi_1 = 3\left(\frac{(4-2)!(2-1)!}{4!}\right)(1-0) + 3\left(\frac{(4-3)!(3-1)!}{4!}\right)(1-0) = \frac{1}{2}.$

$\phi_2 = \phi_3 = \phi_4 = 1\cdot\left(\frac{(4-2)!(2-1)!}{4!}\right)(1-0) + 1\cdot\left(\frac{(4-3)!(3-1)!}{4!}\right)(1-0) = \frac{1}{6}.$

6. By the conditions, we have $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = 0$ and

$$\begin{cases} v(\{1,2\}) = v(\{1,3\}) = v(\{1,4\}) = v(\{2,3\}) = 1 \\ v(\{2,4\}) = v(\{3,4\}) = 0 \\ v(S) = 1 \text{ for any } S \text{ with } |S| \geq 3 \end{cases}$$

Thus $\phi_1 = 3\cdot\frac{(4-2)!(2-1)!}{4!}\cdot 1 + 2\cdot\frac{(4-3)!(3-1)!}{4!}\cdot 1 = \frac{5}{12}$

$\phi_2 = 2\cdot\frac{(4-2)!(2-1)!}{4!}\cdot 1 + 1\cdot\frac{(4-3)!(3-1)!}{4!}\cdot 1 = \frac{1}{4}$

$\phi_3 = 2\cdot\frac{(4-2)!(2-1)!}{4!}\cdot 1 + 1\cdot\frac{(4-3)!(3-1)!}{4!}\cdot 1 = \frac{1}{4}$

$\phi_4 = 1\cdot\frac{(4-2)!(2-1)!}{4!}\cdot 1 + 0\cdot\frac{(4-3)!(3-1)!}{4!}\cdot 1 = \frac{1}{12}$ (or $\phi_4 = 1 - \phi_1 - \phi_2 - \phi_3 = \frac{1}{12}$).

7. Let $A = \{R_1, R_2, L_1, L_2, L_3\}$. Observe that the coalitions containing $R_1$ are $\{R_1\}$, $\{R_1, R_2\}$, $\{R_1, L_1\}^*$, $\{R_1, L_2\}^*$, $\{R_1, L_3\}^*$, $\{R_1, R_2, L_1\}$, $\{R_1, R_2, L_2\}$, $\{R_1, R_2, L_3\}$, $\{R_1, L_1, L_2\}^*$, $\{R_1, L_1, L_3\}^*$, $\{R_1, L_2, L_3\}^*$, $\{R_1, R_2, L_1, L_2\}$, $\{R_1, R_2, L_1, L_3\}^*$, $\{R_1, R_2, L_2, L_3\}^*$, $\{R_1, L_1, L_2, L_3\}^*$, $\{R_1, R_2, L_1, L_2, L_3\}^*$.

Ignore those coalitions above that satisfy $v(S) - v(S\backslash\{R_1\}) = 0$. Consider remaining coalitions with star.

Thus $\phi_{R_1} = \binom{3}{1}\frac{(5-2)!(2-1)!}{5!}(1-0) + \binom{3}{2}\frac{(5-3)!(3-1)!}{5!}(1-0) + \binom{3}{2}\frac{(5-4)!(4-1)!}{5!}(2-1)$

$+ \frac{(5-4)!(4-1)!}{5!}(1-0) + \frac{(5-5)!(5-1)!}{5!}(2-1)$

$= \frac{13}{20}.$

By the symmetry, $\phi_{R_2} = \phi_{R_1} = \frac{13}{20}$. And since $v(\{R_1\}) = v(\{R_2\}) = v(\{L_1\}) = v(\{L_2\}) = v(\{L_3\}) = 0$, $v(\{R_1, R_2, L_1, L_2, L_3\}) = 2$

Hence $\phi_{L_1} = \phi_{L_2} = \phi_{L_3} = \dfrac{2 - 2\times\frac{13}{20}}{3} = \dfrac{7}{30}$.

8. (a) First we have $u(\{1\}) = u(\{2\}) = u(\{3\}) = 0$, $u(\{1,2,3\}) = 1$.

Next we calculate $k = \dfrac{1}{v(\{1,2,3\}) - (v(\{1\}) + v(\{2\}) + v(\{3\}))} = \dfrac{1}{1+a+b+c}$

and we have

$$u(\{1,2\}) = k(v(\{1,2\}) - (v(\{1\}) + v(\{2\}))) = \frac{a+b+c}{1+a+b+c}$$
$$u(\{1,3\}) = k(v(\{1,3\}) - (v(\{1\}) + v(\{3\}))) = \frac{a+b+c}{1+a+b+c}$$
$$u(\{2,3\}) = k(v(\{2,3\}) - (v(\{2\}) + v(\{3\}))) = \frac{a+b+c}{1+a+b+c}$$

(b)(i) Let $\vec{x} = (x_1, x_2, x_3) \in C(v)$ if and only if

$$\begin{cases} x_1 \geq -a,\ x_2 \geq -b,\ x_3 \geq -c \\ x_1 + x_2 \geq c,\ x_1 + x_3 \geq b,\ x_2 + x_3 \geq a \\ x_1 + x_2 + x_3 = 1. \end{cases} \iff \begin{cases} -a \leq x_1 \leq 1-a \\ -b \leq x_2 \leq 1-b \\ -c \leq x_3 \leq 1-c \\ x_1 + x_2 + x_3 = 1 \end{cases} \quad (*)$$

We take $x_1 = 1-a$, $x_2 = 1-b$, $x_3 = 1-c$, which satisfy $(*)$. Since $x_1 + x_2 + x_3 = 3-a-b-c = 3-2 = 1$.

Hence an imputation of $v$ which lies in $C(v)$ is $\vec{x} = (1-a, 1-b, 1-c)$.

(ii) Pf. By $(*)$, we have $x_1 + x_2 \leq 1-a+1-b = 2-a-b$. Since $a+b+c = 2$, then $x_1 + x_2 \leq c$.

Since $x_3 \leq 1-c$, $x_1 + x_2 + x_3 = 1$, then $x_1 + x_2 = 1 - x_3 \geq 1-(1-c) = c$, i.e. $x_1 + x_2 \geq c$.

Hence $x_1 + x_2 = c$, and to make the equality hold, the solution must be $x_1 = 1-a$, $x_2 = 1-b$, so

$x_3 = 1 - x_1 - x_2 = 1-c$.

Therefore, we conclude that $C(v) = \{\vec{x}\}$.

9. (a) Pf. (i) $S = \emptyset$ is trivial, $v(S) = \sum_{k=1}^{n} v_k(S)$.

(ii) If $S \neq \emptyset$, we find the biggest number $j$ in $S$, $1 \leq j \leq n$. Since $C_i$ is strictly increasing,

By the conditions, $v(S) = -\max_{i\in S} C_i = -C_j$

Since $j \in S$, and $R_k = \{k, k+1, \cdots, n\}$, we know $j \in S \cap R_k$, $1 \leq k \leq j$, i.e. $S \cap R_k \neq \emptyset$, if $1 \leq k \leq j$;

$S \cap R_k = \emptyset$, if $j+1 \leq k \leq n$, since $j$ is the biggest number in $S$.

Hence $v_k(S) = \begin{cases} -(C_k - C_{k-1}) & \text{if } 1 \leq k \leq j \\ 0 & \text{if } j+1 \leq k \leq n \end{cases}$

Then $\sum_{k=1}^{n} v_k(S) = \sum_{k=1}^{j} v_k(S) + \sum_{k=j+1}^{n} v_k(S)$

$$= \sum_{k=1}^{j} v_k(S)$$

$$= -(c_1-c_0)-(c_2-c_1)-(c_3-c_2)-\cdots-(c_{j-1}-c_{j-2})-(c_j-c_{j-1})$$

$$= \cancel{-c_1}\cancel{-c_2}\cancel{+c_1}\cancel{-c_3}\cancel{+c_2}-\cdots\cancel{-c_{j-1}}\cancel{+c_{j-2}}-c_j\cancel{+c_{j-1}} \quad (c_0=0)$$

$$= -c_j$$

Hence $v(S) = \sum_{k=1}^{n} v_k(S)$.

Therefore we conclude that $v = \sum_{k=1}^{n} v_k$.

(b) Pf. We need to show $v_k(S\cup\{i\}) = v_k(S)$ for any coalition $S$, $k=1,2,\cdots n$.

Fix $k$, for any coalition $S$,

(i) If $S\cap R_k \neq \emptyset$, then $(S\cup\{i\})\cap R_k \neq \emptyset$.

Hence $v_k(S\cup\{i\}) = v_k(S) = -(c_k-c_{k-1})$

(ii) If $S\cap R_k = \emptyset$, since $i\notin R_k$, then $(S\cup\{i\})\cap R_k = \emptyset$.

Hence $v_k(S\cup\{i\}) = v_k(S) = 0$.

Therefore, the player $i$ is a null player of $v_k$.

(c) Pf. We need to show $v_k(S\cup\{i\}) = v_k(S\cup\{j\})$ for any coalition $S$, $k=1,2,\cdots n$.

Fix $k$, for any coalition $S$,

Since $i,j\in R_k$, then $(S\cup\{i\})\cap R_k\neq\emptyset$, $(S\cup\{j\})\cap R_k\neq\emptyset$.

By the condition, we have $v_k(S\cup\{i\}) = v_k(S\cup\{j\}) = -(c_k-c_{k-1})$.

Therefore, the player $i$ and player $j$ are symmetric players of $v_k$.

(d) By the lecture notes, the Shapley vector $\phi(v)$ has linearity property, and by (a), we have

$\phi_i(v) = \sum_{k=1}^{n}\phi_i(v_k)$

By (b), we know $\phi_i(v_k) = 0$ for all $k>i$.

By (c), we know $\phi_i(v_k) = \phi_k(v_k)$ for all $i\geq k$.

Thus we just need to calculat $\phi_k(v_k)$, $k=1,2,\cdots,n$.

By the condition $v_k(S) = \begin{cases} -(c_k-c_{k-1}) & \text{if } S\cap R_k\neq\emptyset \\ 0 & \text{if } S\cap R_k=\emptyset \end{cases}$, where $R_k=\{k,k+1,\cdots,n\}$.

We put the set $A_k = \{1, 2, \cdots k-1\}$, for any coalition $S$ containing $k$.

To make $\delta_k(S) = v(S) - v(S \setminus \{k\}) \neq 0$, there must be $S \setminus \{k\} \in P(A_k)$, where $P(A_k)$ is the power of $A_k$.

Hence $\phi_k(v_k) = \sum_{S \in P(A_k)} \frac{(n-|S|-1)!\,|S|!}{n!} (v(S \cup \{k\}) - v(S))$

$$= \frac{-C_k + C_{k-1}}{n!} \sum_{S \in P(A_k)} (n-|S|-1)!\,|S|!$$

$$= \frac{-C_k + C_{k-1}}{n!} \sum_{j=0}^{k-1} \binom{k-1}{j} (n-j-1)!\, j! \quad \left( = \frac{-C_k + C_{k-1}}{n-j+1} \right)$$

Therefore $\phi_1(v) = \sum_{k=1}^{n} \phi_1(v_k) = \phi_1(v_1) = \frac{-C_1}{n!} \cdot (n-1)! = \frac{-C_1}{n}$ ;

$$\phi_2(v) = \sum_{k=1}^{n} \phi_2(v_k) = \phi_2(v_1) + \phi_2(v_2) = \phi_1(v_1) + \phi_2(v_2) = \frac{-C_1}{n} + \frac{-C_2 + C_1}{n!}[(n-1)! + (n-2)!]$$

$$= \frac{-C_2}{n-1} + \frac{C_1}{n(n-1)} = \frac{-C_1}{n} + \frac{-C_2 + C_1}{n-1}$$

$$\phi_3(v) = \sum_{k=1}^{n} \phi_3(v_k) = \phi_1(v_1) + \phi_2(v_2) + \phi_3(v_3) = \frac{-C_2}{n-1} + \frac{C_1}{n(n-1)} + \frac{-C_3 + C_2}{n!}[(n-1)! + 2(n-2)! + (n-3)!\,2!]$$

$$= \frac{-C_3}{n-2} + \frac{C_2}{(n-1)(n-2)} + \frac{C_1}{n(n-1)} = \frac{-C_1}{n} + \frac{-C_2 + C_1}{n-1} + \frac{-C_3 + C_2}{n-2}$$

By induction, $\phi_i(v) = \sum_{k=1}^{n} \phi_i(v_k) = \sum_{k=1}^{i} \phi_k(v_k) = \sum_{k=1}^{i} \left[ \frac{-C_k + C_{k-1}}{n!} \sum_{j=0}^{k-1} \binom{k-1}{j} (n-j-1)!\, j! \right]$

$$= \frac{-C_i}{n-i+1} + \frac{C_{i-1}}{(n-i+2)(n-i+1)} + \frac{C_{i-2}}{(n-i+3)(n-i+2)} + \cdots + \frac{C_1}{n(n-1)}$$

$$= \frac{-C_1}{n} + \frac{-C_2 + C_1}{n-1} + \cdots + \frac{-C_i + C_{i-1}}{n-i+1}.$$

Therefore, the Shapley value $\phi_k(v) = \sum_{i=1}^{k} \frac{-C_i + C_{i-1}}{n-i+1}$, $k = 1, 2, \cdots n$.

10. Pf. Let $\sigma$ be the permutation of $A = \{1, 2, \cdots N\}$, there are $N!$ permutations.

View $A$ as the grand coalition, let $S$ be a coalition which contains player $i$.

There are $(|S|-1)!$ number of ways for other players in $S$ to enter the coalition before $i$.

Then player $i$ enters the coalition to form the coalition $S$ and there are $(N-|S|)!$ number of ways for the remaining players to enter to form the grand coalition.

Therefore, among all $N!$ permutations of players in forming the grand coalition, there are $(N-|S|)!(|S|-1)!$ of which the coalition $S$ would form at the moment that player $i$ enters into the coalition.

Thus, we have $\sum_{\{i\} \in S \subset A} (N-|S|)!(|S|-1)! = N!$.